Chapter 8

Applications of Definite Integrals

Section 8.1 Integral as Net Change (pp. 383–393)

Exploration 1 Revisiting Example 2

1. $f'(x)=\dfrac{dy}{dx}=\dfrac{1}{2\sqrt{x+1}}$, and

$$s(0)=\frac{0^3}{3}+\frac{8}{0+1}+C=9\Rightarrow C=1$$

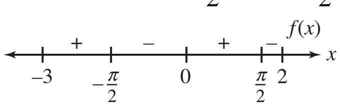
Thus, $s(t)=\dfrac{t^3}{3}+\dfrac{8}{t+1}+1.$

2. $s(1)=\dfrac{1^3}{3}+\dfrac{8}{1+1}+1=\dfrac{16}{3}.$ This is the same as the answer we found in Example 2a.

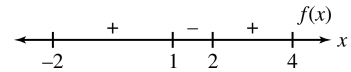
3. $s(5)=\dfrac{5^3}{3}+\dfrac{8}{5+1}+1=44.$ This is the same answer we found in Example 2b.

Quick Review 8.1

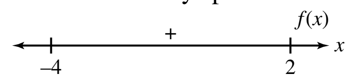
1. On the interval, $\sin 2x = 0$ when $x=-\dfrac{\pi}{2}$, 0, or $\dfrac{\pi}{2}$. Test one point on each subinterval: for $x=-\dfrac{3\pi}{4}$, $\sin 2x=1$; for $x=-\dfrac{\pi}{4}$, $\sin 2x=-1$; for $x=\dfrac{\pi}{4}$, $\sin 2x=1$; and for $x=\dfrac{7\pi}{12}$, $\sin 2x=-\dfrac{1}{2}$. The function changes sign at $-\dfrac{\pi}{2}$, 0, and $\dfrac{\pi}{2}$. The graph is



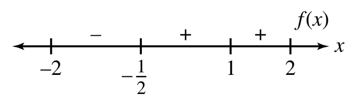
2. $x^2-3x+2=(x-1)(x-2)=0$ when $x=1$ or 2. Test one point on each subinterval: for $x=0$, $x^2-3x+2=2$; for $x=\dfrac{3}{2}$, $x^2-3x+2=-\dfrac{1}{4}$; and for $x=3$, $x^2-3x+2=2$. The function changes sign at 1 and 2. The graph is



3. $x^2-2x+3=0$ has no real solutions, since $b^2-4ac=(-2)^2-4(1)(3)=-8<0$. The function is always positive. The graph is

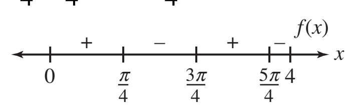


4. $2x^3-3x^2+1=(x-1)^2(2x+1)=0$ when $x=-\dfrac{1}{2}$ or 1. Test one point on each subinterval: for $x=-1$, $2x^3-3x^2+1=-4$; for $x=0$, $2x^3-3x^2+1=1$; and for $x=\dfrac{3}{2}$, $2x^3-3x^2+1=1$. The function changes sign at $-\dfrac{1}{2}$. The graph is



5. On the interval, $x\cos 2x=0$ when $x=0, \dfrac{\pi}{4}, \dfrac{3\pi}{4}$, or $\dfrac{5\pi}{4}$.

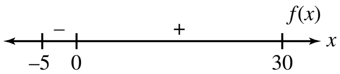
Test one point on each subinterval: for $x=\dfrac{\pi}{8}$, $x\cos 2x=\dfrac{\pi\sqrt{2}}{16}$; for $x=\dfrac{\pi}{2}$, $x\cos 2x=-\dfrac{\pi}{2}$; for $x=\pi$, $x\cos 2x=\pi$; and for $x=4$, $x\cos 2x\approx -0.58$. The function changes sign at $\dfrac{\pi}{4}$, $\dfrac{3\pi}{4}$, and $\dfrac{5\pi}{4}$. The graph is



6. $xe^{-x}=0$ when $x=0$. On the rest of the interval, xe^{-x} is always positive.

7. $\dfrac{x}{x^2+1}=0$ when $x=0$. Test one point on each subinterval: for $x=-1$, $\dfrac{x}{x^2+1}=-\dfrac{1}{2}$; for $x=1$, $\dfrac{x}{x^2+1}=\dfrac{1}{2}$. The function changes sign

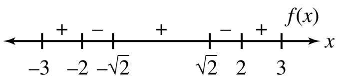
at 0. The graph is



8. $\dfrac{x^2-2}{x^2-4}=0$ when $x=\pm\sqrt{2}$ and is undefined when $x=\pm 2$. Test one point on each subinterval: for $x=-\dfrac{5}{2}$,

$\dfrac{x^2-2}{x^2-4}=\dfrac{17}{9}$; for $x=-1.9$, $\dfrac{x^2-2}{x^2-4}\approx -4.13$; for $x=0$, $\dfrac{x^2-2}{x^2-4}=\dfrac{1}{2}$; for $x=1.9$, $\dfrac{x^2-2}{x^2-4}\approx -4.13$; and for

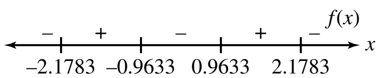
$x=\dfrac{5}{2}$, $\dfrac{x^2-2}{x^2-4}=\dfrac{17}{9}$. The function changes sign at –2, $-\sqrt{2}$, $\sqrt{2}$ and 2. The graph is



9. $\sec\left(1+\sqrt{1-\sin^2 x}\right)=\dfrac{1}{\cos(1+|\cos x|)}$ is undefined when $x\approx 0.9633+k\pi$ or $2.1783+k\pi$ for any integer k.

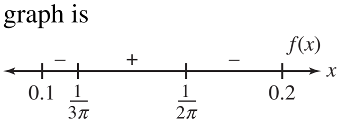
Test for $x=0$: $\sec\left(1+\sqrt{1-\sin^2 0}\right)\approx -2.4030$.

Test for $x=\pm 1$: $\sec\left(1+\sqrt{1-\sin^2 1}\right)\approx 32.7984$. The sign alternates over successive subintervals. The function changes sign at $0.9633+k\pi$ or $2.1783+k\pi$, where k is an integer. The graph is



10. On the interval, $\sin\left(\dfrac{1}{x}\right)=0$ when $\dfrac{1}{3\pi}$ or $\dfrac{1}{2\pi}$. Test one point on each subinterval: for $x=0.1$,

$\sin\left(\dfrac{1}{x}\right)\approx -0.54$; $x=\dfrac{1}{4}y+4$ and for $x=0.2$, $\sin\left(\dfrac{1}{x}\right)\approx -0.96$. The graph changes sign at $\dfrac{1}{3\pi}$, and $\dfrac{1}{2\pi}$. The graph is



Section 8.1 Exercises

1. **(a)** Right when $v(t)>0$, which is when

$\cos t>0$, i.e., when $0\le t<\dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}<t\le 2\pi$. Left when $\cos t<0$, i.e., when $\dfrac{\pi}{2}<t<\dfrac{3\pi}{2}$. Stopped when

$\cos t=0$, i.e., when $t=\dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$.

(b) Displacement $=\int_0^{2\pi}5\cos t\,dt$

$$\begin{aligned}&=5[\sin t]_0^{2\pi}\\&=5[\sin 2\pi-\sin 0]\\&=0\end{aligned}$$

Final position $=3+0=3$

(c) Distance $= \int_0^{2\pi} |5\cos t|\, dt$

$$= \int_0^{\pi/2} 5\cos t\, dt + \int_{\pi/2}^{3\pi/2} -5\cos t\, dt + \int_{3\pi/2}^{2\pi} 5\cos t\, dt$$
$$= 5 + 10 + 5$$
$$= 20$$

2. **(a)** Right when $v(t) > 0$, which is when $\sin 3t > 0$, i.e., when $0 < t < \frac{\pi}{3}$. Left when $\sin 3t < 0$, i.e., when $\frac{\pi}{3} < t \le \frac{\pi}{2}$. Stopped when $\sin 3t = 0$, i.e., when $t = 0$ or $\frac{\pi}{3}$.

(b) Displacement $= \int_0^{\pi/2} 6\sin 3t\, dt$

$$= 6\left[-\frac{1}{3}\cos 3t\right]_0^{\pi/2}$$
$$= -2\left[\cos\frac{3\pi}{2} - \cos 0\right]$$
$$= 2$$

(c) Distance $= \int_0^{\pi/2} |6\sin 3t|\, dt$

$$= \int_0^{\pi/3} 6\sin 3t\, dt + \int_{\pi/3}^{\pi/2} -6\sin 3t\, dt$$
$$= 4 + 2$$
$$= 6$$

3. **(a)** Right when $v(t) = 49 - 9.8t > 0$, i.e., when $0 \le t < 5$.
Left when $49 - 9.8t < 0$, i.e., when $5 < t \le 10$.
Stopped when $49 - 9.8t = 0$, i.e., when $t = 5$.

(b) Displacement $= \int_0^{10} (49 - 9.8t)\, dt$

$$= \left[49t - 4.9t^2\right]_0^{10}$$
$$= 49[(10 - 10) - 0]$$
$$= 0$$

Final position = 3 + 0 = 3

(c) Distance $= \int_0^{10} |49 - 9.8t|\, dt$

$$= \int_0^5 (49 - 9.8t)\, dt + \int_5^{10} (-49 + 9.8t)\, dt$$
$$= 122.5 + 122.5$$
$$= 245$$

4. **(a)** Right when $v(t) = 6t^2 - 18t + 12 = 6(t-1)(t-2) > 0$, i.e., when $0 \le t < 1$.
Left when $6(t-1)(t-2) < 0$, i.e., when $1 < t < 2$. Stopped when $6(t-1)(t-2) = 0$, i.e., when $t = 1$ or 2.

(b) Displacement $= \int_0^2 (6t^2 - 18t + 12)\,dt$

$$= \left[2t^3 - 9t^2 + 12t\right]_0^2$$
$$= [(16 - 36 + 24) - 0]$$
$$= 4$$

Final position = 3 + 4 = 7

(c) Distance = Distance $= \int_0^2 \left|6t^2 - 18t + 12\right| dt$

$$= \int_0^1 (6t^2 - 18t + 12)\,dt + \int_1^2 (-6t^2 + 18t - 12)\,dt$$
$$= 5 + 1$$
$$= 6$$

5. **(a)** Right when $v(t) > 0$, which is when $\sin t \neq 0$ and $\cos t > 0$, i.e., when $0 < t < \frac{\pi}{2}$ or $\frac{3\pi}{2} < t < 2\pi$. Left when $\sin t \neq 0$ and $\cos t < 0$, i.e., when $\frac{\pi}{2} < t < \pi$ or $\pi < t < \frac{3\pi}{2}$. Stopped when $\sin t = 0$ or $\cos t = 0$, i.e., when $t = 0$, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, or 2π.

(b) Displacement $= \int_0^{2\pi} 5\sin^2 t \cos t\,dt$

$$= 5\left[\frac{1}{3}\sin^3 t\right]_0^{2\pi}$$
$$= 5[0 - 0]$$
$$= 0$$

Final position = 3 + 0 = 3

(c) Distance $= \int_0^{2\pi} \left|5\sin^2 t \cos t\right| dt$

$$= \int_0^{\pi/2} 5\sin^2 t \cos t\,dt + \int_{\pi/2}^{3\pi/2} -5\sin^2 t \cos t\,dt + \int_{3\pi/2}^{2\pi} 5\sin^2 t \cos t\,dt$$
$$= \frac{5}{3} + \frac{10}{3} + \frac{5}{3}$$
$$= \frac{20}{3}$$

6. **(a)** Right when $v(t) > 0$, which is when $4 - t > 0$, i.e., when $0 \le t < 4$. Left: never, since $\sqrt{4-t}$ cannot be negative. Stopped when $4 - t = 0$, i.e., when $t = 4$.

(b) Displacement $= \int_0^4 \sqrt{4-t}\,dt = \left[-\frac{2}{3}(4-t)^{3/2}\right]_0^4 = -\frac{2}{3}[0 - 8] = \frac{16}{3}$

Final position $= 3 + \frac{16}{3} = \frac{25}{3}$

(c) Distance $= \int_0^4 \sqrt{4-t}\,dt = \frac{16}{3}$

7. **(a)** Right when $v(t) > 0$, which is when $\cos t > 0$, i.e., when $0 \le t < \frac{\pi}{2}$ or $\frac{3\pi}{2} < t \le 2\pi$. Left when $\cos t < 0$, i.e., when $\frac{\pi}{2} < t < \frac{3\pi}{2}$. Stopped when $\cos t = 0$, i.e., when $t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

(b) Displacement $= \int_0^{2\pi} e^{\sin t}\cos t\,dt = \left[e^{\sin t}\right]_0^{2\pi} = [e^0 - e^0] = 0$

Final position = 3 + 0 = 3

(c) Distance $= \int_0^{2\pi} \left|e^{\sin t}\cos t\right| dt = \int_0^{\pi/2} e^{\sin t}\cos t\,dt + \int_{\pi/2}^{3\pi/2} -e^{\sin t}\cos t\,dt + \int_{3\pi/2}^{2\pi} e^{\sin t}\cos t\,dt$

$$= (e-1) + \left(e - \frac{1}{e}\right) + \left(1 - \frac{1}{e}\right)$$
$$= 2e - \frac{2}{e} \approx 4.7$$

8. **(a)** Right when $v(t) > 0$, which is when $0 < t \le 3$. Left: never, since $v(t)$ is never negative. Stopped when $t = 0$.

(b) Displacement $= \int_0^3 \frac{t}{1+t^2}\,dt$

$$= \left[\frac{1}{2}\ln(1+t^2)\right]_0^3$$
$$= \frac{1}{2}\left[\ln(10) - 1\mathrm{n}(1)\right]$$
$$= \frac{\ln 10}{2} \approx 1.15$$

Final position $= 3 + \frac{\ln 10}{2} \approx 4.15$

(c) Distance $= \int_0^3 \frac{t}{1+t^2}\,dt = \frac{\ln 10}{2} \approx 1.15$

9. **(a)** $v(t) = \int a(t)\,dt = t + 2t^{3/2} + C$, and since $v(0) = 0$, $v(t) = t = +2t^{3/2}$. Then $v(9) = 9 + 2(27) = 63$ mph.

(b) First convert units: $t + 2t^{3/2}$ mph $= \frac{t}{3600} + \frac{t^{3/2}}{1800}$ mi / sec. Then

Distance $= \int_0^9 \left(\frac{t}{3600} + \frac{t^{3/2}}{1800}\right) dt$

$$= \left[\frac{t^2}{7200} + \frac{t^{5/2}}{4500}\right]_0^9$$
$$= \left[\left(\frac{9}{800} + \frac{27}{500}\right) - 0\right]$$
$$= 0.06525\ \text{mi}$$
$$= 344.52\ \text{ft.}$$

10. **(a)** Displacement $= \int_0^4 (t-2)\sin t\,dt$

$$= \left[\sin t - t\cos t + 2\cos t\right]_0^4$$
$$= \left[(\sin 4 - 4\cos 4 + 2\cos 4) - 2\right]$$
$$\approx -1.44952\,\text{m}$$

(b) Because the velocity is negative for $0 < t < 2$, positive for $2 < t < \pi$, and negative for $\pi < t \le 4$,

Distance $= \int_0^2 -(t-2)\sin t\,dt + \int_2^{\pi} (t-2)\sin t\,dt + \int_{\pi}^4 -(t-2)\sin t\,dt$

$$= [(2-\sin 2) + (\pi - \sin 2 - 2) + (\pi + 2\cos 4 - \sin 4 - 2]$$
$$= 2\pi + 2\cos 4 - 2\sin 2 - \sin 4 - 2 \approx 1.914\,\text{m}.$$

11. **(a)** $v(t) = \int a(t)\,dt = \int -32\,dt = -32t + C_1$, where $C_1 = v(0) = 90$. Then $v(3) = -32(3) + 90 = -6$ ft/sec.

(b) $s(t) = \int v(t)\,dt = -16t^2 + 90t + C_2$, where $C_2 = s(0) = 0$. Solve $s(t) = 0$: $-16t^2 + 90t = 2t(-8t + 45) = 0$ when $t = 0$ or $t = \frac{45}{8} = 5.625$ sec.

The projectile hits the ground at 5.625 sec.

(c) Since starting height = ending height, Displacement = 0.

(d) Max. Height $= s\left(\frac{5.625}{2}\right)$

$$= -16\left(\frac{5.625}{2}\right)^2 + 90\left(\frac{5.625}{2}\right)$$
$$= 126.5625,$$

and Distance = 2(Max. Height) = 253.125 ft.

12. Displacement $= \int_0^c v(t)\,dt$

$$= -4 + 5 - 24$$
$$= -23\,\text{cm}$$

13. Total distance $= \int_0^c |v(t)|\,dt$

$$= 4 + 5 + 24$$
$$= 33\,\text{cm}$$

14. At $t = a$, $s = s(0) + \int_0^a v(t)\,dt = 15 - 4 = 11.$

At $t = b$, $s = s(0) + \int_0^b v(t)\,dt = 15 - 4 + 5 = 16.$

At $t = c$, $s = s(0) + \int_0^c v(t)\,dt = 15 - 4 + 5 - 24 = -8.$

15. At $t = a$, where $\frac{dv}{dt}$ is at a maximum (the graph is steepest upward).

16. At $t = c$, where $\frac{dv}{dt}$ is at a maximum (the graph is steepest upward).

17. Distance = Area under curve

$$\begin{aligned} &= 4\left(\frac{1}{2}\cdot 1\cdot 2\right) \\ &= 4 \end{aligned}$$

(a) Final position = Initial position + Distance

$$\begin{aligned} &= 2+4 \\ &= 6;\ \text{ends at } x = 6. \end{aligned}$$

(b) 4 meters

18. **(a)** Positive and negative velocities cancel; the sum of signed areas is zero. Starts and ends at $x = 2$.

(b) Distance = Sum of positive areas

$$\begin{aligned} &= 4(1\cdot 1) \\ &= 4\ \text{meters} \end{aligned}$$

19. **(a)** Final position $= 2 + \int_0^7 v(t)\,dt$

$$\begin{aligned} &= 2 - \frac{1}{2}(1)(2) + \frac{1}{2}(1)(2) + 1(2) + \frac{1}{2}(2)(2) - \frac{1}{2}(2)(1) \\ &= 5; \end{aligned}$$

end at $x = 5$.

(b) $\int_0^7 |v(t)|\,dt = \frac{1}{2}(1)(2) + \frac{1}{2}(1)(2) + 1(2) + \frac{1}{2}(2)(2) + \frac{1}{2}(2)(1)$

$= 7$ meters

20. **(a)** Final position $= 2 + \int_0^{10} v(t)$

$$\begin{aligned} &= 2 + \frac{1}{2}(2)(3) - \frac{1}{2}(1)(3) - (3)(3) - \frac{1}{2}(1)(3) + \frac{1}{2}(3)(3) \\ &= -2.5; \end{aligned}$$

ends at $x = -2.5$

(b) Distance $= \int_0^{10} |v(t)|\,dt$

$$\begin{aligned} &= \frac{1}{2}(2\cdot 3) + \frac{1}{2}(1)(3) + 3(3) + \frac{1}{2}(1)(3) + \frac{1}{2}(3) \\ &= 19.5\ \text{meters} \end{aligned}$$

21. $\int_0^{10} 27.08 \cdot e^{t/25}\,dt = 27.08[25e^{t/25}]_0^{10}$

$$\begin{aligned} &= 27.08[25e^{0.4} - 25] \\ &\approx 332.965\ \text{billion barrels} \end{aligned}$$

22. $\int_0^{24}\left[3.9-2.4\sin\left(\frac{\pi t}{12}\right)\right]dt$

$=\left[3.9t+\frac{28.8}{\pi}\cos\left(\frac{\pi t}{12}\right)\right]_0^{24}$

$=\left[\left(93.6+\frac{28.8}{\pi}\right)-\frac{28.8}{\pi}\right]dt$

$=93.6$ kilowatt-hours

23. **(a)** Solve $10{,}000(2-r)=0$: $r=2$ miles.

(b) Width $=\Delta r$; Length $=2\pi r$;
Area $=2\pi r\Delta r$

(c) Population = Population density × Area

(d) $\int_0^2 10{,}000(2-r)(2\pi r)\,dr$

$=20{,}000\pi\int_0^2(2r-r^2)\,dr$

$=20{,}000\pi\left[r^2-\frac{1}{3}r^3\right]_0^2$

$=20{,}000\pi\left[\left(4-\frac{8}{3}\right)-0\right]$

$=\frac{80{,}000}{3}\pi\approx 83{,}776$

24. **(a)** Width $=\Delta r$, Length $=2\pi r$:
Area $=2\pi r\Delta r$

(b) Volume per second
= Inches per second × Cross section area

$8(10-r^2)\frac{\text{in.}}{\text{sec}}\cdot(2\pi r)\Delta r\,\text{in}^2$

$=\text{flow in}\,\frac{\text{in}^3}{\text{sec}}$

(c) $\int_0^3 8(10-r^2)(2\pi r)dr$

$=16\pi\int_0^3(10r-r^3)\,dr$

$=16\pi\left[5r^2-\frac{1}{4}r^4\right]_0^3$

$=16\pi\left[\left(45-\frac{81}{4}\right)-0\right]$

$=396\pi\frac{\text{in}^3}{\text{sec}}\approx 1244.07\frac{\text{in}^3}{\text{sec}}$

25. **(a)** Sum of numbers in Sales column
= 797.5 thousand

(b) Enter the table in a graphing calculator and use QuadReg:
$B(x)=1.6x^2+2.3x+5.0.$

(c) $\int_0^{11}(1.6x^2+2.3x+5.0)\,dx$

$=\left[\frac{1.6}{3}x^3+\frac{2.3}{2}x^2+5.0x\right]_0^{11}$

≈ 904.02 thousand

(d) The answer in (a) corresponds to the area of left hand rectangles. These rectangles lie under the curve $B(x)$. The answer in (c) corresponds to the area under the curve. This area is greater than the area of the rectangles.

26. **(a)** $\int_{-0.5}^{10.5}\left(1.6x^2+2.3x+5.0\right)dr$

$=\left[\frac{1.6}{3}x^3+\frac{2.3}{2}x^2+5.0x\right]_{-0.5}^{10.5}$

≈ 798.97 thousand

(b) The answer in (a) corresponds to the area of rectangles whose heights are the actual sales ("midpoint rectangles"). The curve now gives a better approximation since part of each rectangle is above the curve and part is below.

27. Treat 6 P.M. as 18 o'clock:

$$\frac{b-a}{2n}\left[f(x_0)+\sum_{i=1}^{n-1}2f(x_i)+f(x_n)\right]$$
$$=\frac{18-8}{2(10)}[120+2(110)+2(115)+2(119)+2(120)+2(120)+2(115)+2(112)+2(110)+(121)]$$
$$=1156.5$$

28. (Answer may vary.)
Plot the speeds vs. time. Connect the points and find the area under the line graph. The definite integral also gives the area under the curve.

29. $F(x) = kx$; $6 = k(3)$, so $k = 2$ and $F(x) = 2x$.

(a) $F(9) = 2(9) = 18\text{N}$

(b)
$$\begin{aligned} W &= \int_0^9 F(x)\,dx \\ &= \int_0^9 2x\,dx \\ &= [x^2]_0^9 \\ &= 81\,\text{N}\cdot\text{cm} \end{aligned}$$

30. $F(x) = kx$; $10{,}000 = k(1)$, so $k = 10{,}000$.

(a)
$$\begin{aligned} W &= \int_0^d kx\,dx \\ &= \left[\frac{1}{2}kx^2\right]_0^d \\ &= \frac{1}{2}kd^2 \\ &= \frac{1}{2}(10{,}000)(0.5)^2 \\ &= 1250\,\text{inch-pounds} \end{aligned}$$

(b) For total distance: $W = \frac{1}{2}(10{,}000)(1)^2 = 5000$

For second half of distance: $W = 5000 - 1250 = 3750$ inch-pounds

31. False; the displacement is the integral of the velocity from $t = 0$ to $t = 5$ and is positive, since the region that is under the graph and above the horizontal axis is larger than the region that is above the graph and below the horizontal axis.

32. True; since the velocity is positive, the integral of the velocity is equal to the integral of its absolute value, which is the total distance traveled.

33. C; to the nearest whole square, the area under the curve covers 12 grid squares. $(12)(50)(6) = 3600$.

34. D; $5+\frac{15}{10}(4+2(8)+2(6)+2(9)+2(10)+10)=125$.

35. B; $\int_0^{60}\left(12+6\cos\left(\frac{t}{\pi}\right)\right)dt = 12t+6\pi\sin\left(\frac{t}{\pi}\right)\Big|_0^{60}$
$\approx 725.$

36. A; $\int_0^{10} 20e^{-0.5t}\,dt = -40e^{-0.5t}\Big|_0^{10} = 40$

37. $\frac{(12-0)}{2(12)}[0.04+2(0.04)+2(0.05)+2(0.06)+2(0.05)+2(0.04)+2(0.05)+2(0.04)+2(0.06)+2(0.05)+0.05]$
$=0.585$

The overall rate, then, is $\frac{0.585}{12} = 0.04875.$

38. $\frac{(12-0)}{2(12)}[3.6+2(4.0)+2(3.1)+2(2.8)+2(2.8)+2(3.2)+2(3.3)+2(3.1)+2(3.2)+2(3.4)+2(3.4)+2(3.9)+4.0]$
$= 40$ thousandths or 0.040

39. **(a)** $\bar{x} = \frac{M_y}{M} = \frac{\sum m_k x_k}{\sum m_k}$. Taking $dm = \delta\, dA$ as m_k and letting $dA \to 0$, $k \to \infty$ yields $\frac{\int x\,dm}{\int dm}$.

(b) $\bar{y} = \frac{M_k}{M} = \frac{\sum m_k y}{\sum m_k}$. Taking $dm = \delta\, dA$ as m_k and letting $dA \to 0$, $k \to \infty$ yields $\frac{\int y\,dm}{\int dm}$.

40. By symmetry, $\bar{x} = 0$. For $\bar{y}$, use horizontal strips:

$$\bar{y} = \frac{\int y\,dm}{\int dm} = \frac{\int y\,\delta\,dA}{\int \delta\,dA} = \frac{\int y\,dA}{\int dA} = \frac{\int_0^4 y\left(2\sqrt{y}\right)dy}{\int_0^4 2\sqrt{y}\,dy} = \frac{2\left[\frac{2}{5}y^{5/2}\right]_0^4}{2\left[\frac{2}{3}y^{3/2}\right]_0^4} = \frac{12}{5}$$

41. By symmetry, $\bar{y} = 0$. For $\bar{x}$, use vertical strips:

$$\bar{x} = \frac{\int x\,dm}{\int dm} = \frac{\int x\delta\,dA}{\int \delta\,dA} = \frac{\int x\,dA}{\int dA} = \frac{\int_0^2 x(2x)\,dx}{\int_0^2 2x\,dx} = \frac{\left[\frac{2}{3}x^3\right]_0^2}{[x^2]_0^2} = \frac{4}{3}$$

Section 8.2 Areas in the Plane (pp. 394–402)

Exploration 1 A Family of Butterflies

1. For $k = 1$:

$$\int_0^{\pi} [(2 - \sin x) - \sin x]\,dx = \int_0^{\pi} (2 - 2\sin x)\,dx = [2x + 2\cos x]_0^{\pi} = 2\pi - 4$$

For $k = 2$:

$$\int_0^{\pi/2} [(4 - 2\sin 2x) - (2\sin 2x)]\,dx = \int_0^{\pi/2} (4 - 4\sin 2x)\,dx = [4x + 2\cos 2x]_0^{\pi/2} = 2\pi - 4$$

2. It appears that the areas for $k \ge 3$ will continue to be $2\pi - 4$.

3. $$A_k = \int_0^{\pi/k} [(2k - k\sin kx) - k\sin kx]\,dx = \int_0^{\pi/k} [(2k - 2k\sin kx)\,dx$$

If we make the substitution $u = kx$, then $du = k\,dx$ and the u-limits become 0 to π Thus,

$$A_k = \int_0^{\pi/k} [(2k - 2k\sin kx)\,dx = \int_0^{\pi/k} [(2 - 2\sin kx)k\,dx = \int_0^{\pi} (2 - 2\sin u)\,du.$$

4. $2\pi - 4$

5. Because the amplitudes of the sine curves are k, the kth butterfly stands $2k$ units tall. The vertical edges alone have lengths $(2k)$ that increase without bound, so the perimeters are tending to infinity.

Quick Review 8.2

1. $\int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} = -[-1 - 1] = 2$

2. $\int_0^1 e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_0^1 = \frac{1}{2}(e^2 - 1)$

3. $\int_{-\pi/4}^{\pi/4} \sec^2 x\,dx = [\tan x]_{-\pi/4}^{\pi/4} = 1 - (-1) = 2$

4. $$\int_0^2 (4x - x^3)\,dx = \left[2x^2 - \frac{1}{4}x^4\right]_0^2 = (8 - 4) - 0 = 4$$

5. $\int_{-3}^{3} \sqrt{9 - x^2}\,dx = \frac{9\pi}{2}$ (This is half the area of a circle of radius 3.)

6. Solve $x^2 - 4x = x + 6$.

$x^2 - 5x - 6 = 0$
$(x - 6)(x + 1) = 0$
$x = 6$ or $x = -1$
$y = 6 + 6 = 12$ or $y = -1 + 6 = 5$
(6, 12) and (–1, 5)

7. Solve $e^x = x + 1$. From the graphs, it appears that e^x is always greater than or equal to $x + 1$, so that if they are ever equal, this is when $e^x - (x + 1)$ is at a minimum.

$\frac{d}{dx}[e^x - (x + 1)] = e^x - 1$ is zero when $e^x = 1$, i.e., when $x = 0$. Test: $e^0 = 0 + 1 = 1$. So the solution is (0, 1).

8. Inspection of the graphs shows two intersection points: (0, 0), and $(\pi, 0)$. Check: $0^2 - \pi \cdot 0 = \sin 0 = 0$ and $\pi^2 - \pi^2 = \sin \pi = 0$.

9. Solve $\frac{2x}{x^2+1} = x^3$.

(0, 0) is a solution. Now divide by x.

$$\frac{2}{x^2+1} = x^2$$
$$2 = x^4 + x^2$$
$$x^4 + x^2 - 2 = 0$$
$$x^2 = \frac{-1 \pm \sqrt{1+8}}{2} = -2 \text{ or } 1$$

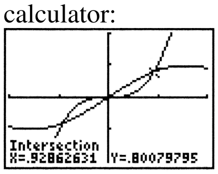
Throw out the negative solution.

$x = \pm 1$

$y = x^3 = \pm 1$

(0, 0), (–1, –1) and (1, 1)

10. Use the intersect function on a graphing calculator:



[–2, 2] by [–2, 2]

(–0.9286, –0.8008), (0, 0), and (0.9286, 0.8008)

Section 8.2 Exercises

1. $\int_0^{\pi} (1-\cos^2 x)\,dx = \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi} = \frac{\pi}{2}$

2. Use symmetry:

$$\int_{-\pi/3}^{\pi/3} \left(\frac{1}{2}\sec^2 t + 4\sin^2 t\right) dt$$
$$= \int_0^{\pi/3} (\sec^2 + 8\sin^2 t)\,dt$$
$$= [\tan t + 4t - 2\sin 2t]_0^{\pi/3}$$
$$= \left(\sqrt{3} + \frac{4\pi}{3} - \sqrt{3}\right) - 0$$
$$= \frac{4\pi}{3}$$

3. $\int_0^1 (y^2 - y^3)\,dy = \left[\frac{1}{3}y^3 - \frac{1}{4}y^4\right]_0^1 = \frac{1}{12}$

4. $F'(x) = \sqrt{x^4 - 1}$,

$$\frac{d}{dx}\left(1 + \sqrt[3]{\sin x}\right) = \frac{\cos x}{3(\sin x)^{2/3}}$$
$$= \left[-3y^4 + \frac{10}{3}y^3 + y^2\right]_0^1$$
$$= -3 + \frac{10}{3} + 1$$
$$= \frac{4}{3}$$

5. Use the region's symmetry:

$$2\int_0^2 [2x^2 - (x^4 - 2x^2)]\,dx$$
$$= 2\int_0^2 (-x^4 + 4x^2)\,dx$$
$$= 2\left[-\frac{1}{5}x^5 + \frac{4}{3}x^3\right]_0^2$$
$$= 2\left[\left(-\frac{32}{5} + \frac{32}{3}\right) - 0\right]$$
$$= \frac{128}{15}$$

6. Use the region's symmetry:

$$2\int_0^1 (x^2 + 2x^4)\,dx = 2\left[\frac{1}{3}x^3 + \frac{2}{5}x^5\right]_0^1$$
$$= 2\left(\frac{1}{3} + \frac{2}{5}\right)$$
$$= \frac{22}{15}$$

7. The functions intersect at $x \approx -1.4096$ and $x \approx 0.6367$.

$$\int_{-1.4096}^{0.6367} [(1-x^2) - \sin x]dx \approx 1.670.$$

8. The functions intersect at $x \approx \pm 1.152961$.

$$\int_{-1.152961}^{1.152961} [\cos 2x - (x^2 - 2)]dx \approx 4.332.$$

9. $\int_0^1 \left(x - \frac{x^2}{4}\right)dx + \int_1^2 \left(1 - \frac{x^2}{4}\right)dx$

$$= \left(\frac{x^2}{2} - \frac{x^3}{12}\right)\Bigg|_0^1 + \left(x - \frac{x^3}{12}\right)\Bigg|_1^2$$
$$= \left(\frac{1^2}{2} - \frac{1^3}{12}\right) - \left(\frac{0^2}{12} - \frac{0^3}{12}\right) + \left(2 - \frac{2^3}{12}\right) - \left(1 - \frac{1^3}{12}\right)$$
$$= \frac{5}{6}$$

10. $\int_0^1 x^2 dx + \int_1^2 (-x+2)\, dx = \left.\dfrac{x^3}{3}\right|_0^1 + \left.\left(-\dfrac{x^2}{2}+2x\right)\right|_1^2$

$$= \frac{1^3}{3} - 0 + \left(-\frac{2^2}{2} + 2(2)\right) - \left(-\frac{1^2}{2} + 2(1)\right)$$

$$= \frac{5}{6}$$

11. $\int_{-\sqrt{2}}^{\sqrt{2}} (3 - y^2 - (y^2 - 1))\, dy = \int_{-\sqrt{2}}^{\sqrt{2}} (4 - 2y^2)\, dy$

$$= \left.\left(4y - \frac{2y^3}{3}\right)\right|_{-\sqrt{2}}^{\sqrt{2}}$$

$$= 4\sqrt{2} - \frac{2\left(\sqrt{2}\right)^3}{3} - \left(4\left(-\sqrt{2}\right) - \frac{2\left(-\sqrt{2}\right)^3}{3}\right)$$

$$= \frac{16\sqrt{2}}{3} \approx 7.542$$

12. $\int_{-3/2}^{2} \left(\dfrac{y}{2} - (y^2 - 3)\right) dy = \left.\left(\dfrac{y^2}{4} - \dfrac{y^3}{3} + 3y\right)\right|_{-3/2}^{2}$

$$= \frac{(2)^2}{4} - \frac{(2)^3}{3} + 3(2) - \left(\frac{\left(-\frac{3}{2}\right)^2}{4} - \frac{\left(-\frac{3}{2}\right)^3}{3} + 3\left(-\frac{3}{2}\right)\right)$$

$$= 7\frac{7}{48} \approx 7.146$$

13. $\int_{-2}^{0} ((2x^3 - x^2 - 5x) - (-x^2 + 3x))\, dx + \int_0^2 (-x^2 + 3x - (2x^3 - x^2 - 5x))\, dx$

$$= \int_{-2}^{0} (2x^3 - 8x) dx + \int_0^2 (-2x^3 + 8x) dx$$

$$= \left.\left(\frac{x^4}{2} - 4x^2\right)\right|_{-2}^{0} + \left.\left(-\frac{x^4}{2} + 4x^2\right)\right|_0^2$$

$$= \left(0 - \left(\frac{(-2)^4}{2} - 4(-2)^2\right)\right) + \left(\left(-\frac{2^4}{2} + 4(2)^2\right) - 0\right)$$

$$= \left(0 - \left(\frac{16}{2} - 16\right)\right) + \left(\left(-\frac{16}{2} + 16\right) - 0\right)$$

$$= 16$$

14. $\int_{-2}^{-1}(-x+2-(4-x^2))\,dx+\int_{-1}^{2}(4-x^2-(-x+2))\,dx+\int_{2}^{3}(-x+2)-(4-x^2)\,dx$

$=\int_{-2}^{-1}(-x-2+x^2)dx+\int_{-1}^{2}(x+2-x^2)dx+\int_{2}^{3}(-x-2+x^2)dx$

$=\left(-\frac{x^2}{2}-2x+\frac{x^3}{3}\right)\Bigg|_{-2}^{-1}+\left(\frac{x^2}{2}+2x-\frac{x^3}{3}\right)\Bigg|_{-1}^{2}+\left(-\frac{x^2}{2}-2x+\frac{x^3}{3}\right)\Bigg|_{2}^{3}$

$=\left(-\frac{(-1)^2}{2}-2(-1)+\frac{(-1)^3}{3}-\left(-\frac{(-2)^2}{2}-2(-2)+\frac{(-2)^3}{3}\right)\right)+\left(\frac{2^2}{2}+2(2)-\frac{2^3}{3}-\left(\frac{(-1)^2}{2}+2(-1)-\frac{(-1)^3}{3}\right)\right)$

$+\left(-\frac{3^2}{2}-2(3)+\frac{3^3}{3}-\left(-\frac{2^2}{2}-2(2)+\frac{2^3}{3}\right)\right)$

$=8\frac{1}{6}$

15. Solve $x^2-2=2$: $x^2=4$, so the curves intersect at $x=\pm 2$.

$$\begin{aligned}\int_{-2}^{2}[2-(x^2-2)]\,dx&=\int_{-2}^{2}(4-x^2)\,dx\\&=\left[4x-\frac{1}{3}x^3\right]_{-2}^{2}\\&=\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)\\&=\frac{32}{3}\\&=10\frac{2}{3}\end{aligned}$$

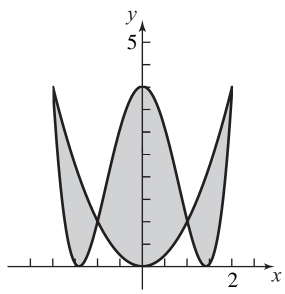
16. Solve $2x-x^2=-3$: $x^2-2x-3=(x-3)(x+1)=0$, so the curves intersect at $x=-1$ and $x=3$.

$$\begin{aligned}\int_{-1}^{3}(2x-x^2+3)\,dx&=\left[x^2-\frac{1}{3}x^3+3x\right]_{-1}^{3}\\&=(9-9+9)-\left(1+\frac{1}{3}-3\right)\\&=\frac{32}{3}\\&=10\frac{2}{3}\end{aligned}$$

17. Solve $7-2x^2=x^2+4$: $x^2=1$, so the curves intersect at $x=\pm 1$.

$$\begin{aligned}\int_{-1}^{1}\left[(7-2x^2)-(x^2+4)\right]dx&=\int_{-1}^{1}(-3x^2+3)\,dx\\&=3\int_{-1}^{1}(1-x^2)\,dx\\&=3\left[x-\frac{1}{3}x^3\right]_{-1}^{1}\\&=3\left[\frac{2}{3}-\left(-\frac{2}{3}\right)\right]\\&=4\end{aligned}$$

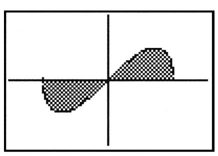
18.



Solve $x^4 - 4x^2 + 4 = x^2$: $x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4) = 0$, so the curves intersect at $x = \pm 1, \pm 2$. Use the region's symmetry:

$$2\int_0^1 \left[\left(x^4 - 4x^2 + 4\right) - x^2\right] dx + 2\int_1^2 \left[x^2 - \left(x^4 - 4x^2 + 4\right)\right] dx$$
$$= 2\int_0^1 (x^4 - 5x^2 + 4)\,dx + 2\int_1^2 (-x^4 + 5x^2 - 4)\,dx$$
$$= 2\left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x\right]_0^1 + 2\left[-\frac{1}{5}x^5 + \frac{5}{3}x^3 - 4x\right]_1^2$$
$$= 2\left[\frac{1}{5} - \frac{5}{3} + 4\right] + 2\left[\left(-\frac{32}{5} + \frac{40}{3} - 8\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right)\right]$$
$$= 8$$

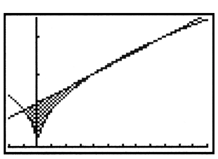
19.



$\left[-\frac{3}{2}a, \frac{3}{2}a\right]$ by $[-a^2, a^2]$

The curves intersect at $x = 0$ and $x = \pm a$. Use the region's symmetry: $\int_0^{0.739} (\cos x - x)\,dx$.

20.



[−2, 12] by [0, 3.5]

The curves intersect at three points: $x = -1$, $x = 4$ and $x = 9$.

Because of the absolute value sign, break the integral up at $x=0$ also:

$$\int_{-1}^{0}\left(\frac{x+6}{5}-\sqrt{-x}\right)dx+\int_{0}^{4}\left(\frac{x+6}{5}-\sqrt{x}\right)dx+\int_{4}^{9}\left(\sqrt{x}-\frac{x+6}{5}\right)dx$$

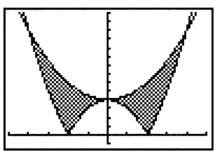
$$=\left[\frac{\frac{1}{2}x^2+6x}{5}+\frac{2}{3}(-x)^{3/2}\right]_{-1}^{0}+\left[\frac{\frac{1}{2}x^2+6x}{5}-\frac{2}{3}x^{3/2}\right]_{0}^{4}+\left[\frac{2}{3}x^{3/2}-\frac{\frac{1}{2}x^2+6x}{5}\right]_{4}^{9}$$

$$=\left[0-\left(-\frac{11}{10}+\frac{2}{3}\right)\right]+\left[\left(\frac{32}{5}-\frac{16}{3}\right)-0\right]+\left[\left(18-\frac{189}{10}\right)-\left(\frac{16}{3}-\frac{32}{5}\right)\right]$$

$$=\frac{13}{30}+\frac{16}{15}+\frac{1}{6}$$

$$=\frac{5}{3}$$

$$=1\frac{2}{3}$$

21. 

[–5, 5] by [–1, 14]

The curves intersect at $x = 0$ and $x = \pm 4$. Because of the absolute value sign, break the integral up at $x=\pm 2$ also (where $\left|x^2-4\right|$ turns the corner). Use the graph's symmetry: $(x)=\pi r^2=\pi\left(\frac{w}{2}\right)^2=\pi(1-x^2)^2.$

22. Solve $y^2=y+2$: $y^2-y-2=(y-2)(y+1)=0,$ so the curves intersect at $y=-1$ and $y=2$.

$$\int_{-1}^{2}(y+2-y^2)\,dy=\left[\frac{1}{2}y^2+2y-\frac{1}{3}y^3\right]_{-1}^{2}$$

$$=\left(2+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)$$

$$=\frac{9}{2}=4\frac{1}{2}$$

23. Solve for x: $x=\frac{y^2}{4}-1$ and $x=\frac{y}{4}+4.$

Now solve $\frac{y^2}{4}-1=\frac{y}{4}+4$: $\frac{y^2}{4}-\frac{y}{4}-5=0,$

$y^2-y-20=(y-5)(y+4)=0.$

The curves intersect at $y=-4$ and $y=5$.

$$\int_{-4}^{5}\left[\left(\frac{y}{4}+4\right)-\left(\frac{y^2}{4}-1\right)\right]dy$$
$$=\int_{-4}^{5}\left(-\frac{y^2}{4}+\frac{y}{4}+5\right)dy$$
$$=\left[-\frac{y^3}{12}+\frac{y^2}{8}+5y\right]_{-4}^{5}$$
$$=\left(-\frac{125}{12}+\frac{25}{8}+25\right)-\left(\frac{16}{3}+2-20\right)$$
$$=\frac{243}{8}$$
$$=30\frac{3}{8}$$

24. Solve for x: $x=y^2$ and $x=3-2y^2$. Now solve $y^2=3-2y^2$: $y^2=1$, so the curves intersect at $y=\pm 1$.
Use the region's symmetry:

$$2\int_0^1(3-2y^2-y^2)\,dy=2\int_0^1(3-3y^2)\,dy$$
$$=6\int_0^1(1-y^2)\,dy$$
$$=6\left[y-\frac{1}{3}y^3\right]_0^1$$
$$=6\left[\left(1-\frac{1}{3}\right)-0\right]$$
$$=4$$

25. Solve for x: $x=-y^2$ and $x=2-3y^2$. Now solve $-y^2=2-3y^2$: $y^2=1$, so the curves intersect at $y=\pm 1$. Use the region's symmetry:

$$2\int_0^1(2-3y^2+y^2)\,dy=2\int_0^1(2-2y^2)\,dy$$
$$=4\int_0^1(1-y^2)\,dy$$
$$=4\left[y-\frac{1}{3}y^3\right]_0^1$$
$$=4\left[\left(1-\frac{1}{3}\right)-0\right]$$
$$=\frac{8}{3}$$

26. Solve for y: $y=4-4x^2$ and $A(x)=\pi r^2=\pi x^4$. Now solve $4-4x^2=x^4-1$:
$x^4+4x^2-5=(x^2-1)(x^2+5)=0$.
The curves intersect at $x=\pm 1$.
Use the region's symmetry:

$$A(x)=\pi r^2$$
$$=\pi(x-x^2)^2$$
$$=\pi(x^2-2x^3+x^4)$$
$$=2\left[-\frac{1}{5}x^5-\frac{4}{3}x^3+5x\right]_0^1$$
$$=2\left[\left(-\frac{1}{5}-\frac{4}{3}+5\right)-0\right]$$
$$=\frac{104}{15}$$
$$=6\frac{14}{15}$$

27. Solve for x: $x=3-y^2$ and $x=-\frac{y^2}{4}$.

Now solve $3-y^2=-\frac{y^2}{4}$: $y^2=4$,
so the curves intersect at $y=\pm 2$.
Use the region's symmetry:

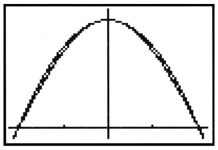
$$2\int_0^2\left(3-y^2+\frac{y^2}{4}\right)dy=2\int_0^2\left(3-\frac{3y^2}{4}\right)dy$$
$$=2\left[3y-\frac{y^3}{4}\right]_0^2$$
$$=2(6-2)-0$$
$$=8$$

28. The curves intersect at 0 and π, so the area is:

$$\int_0^{\pi}(2\sin x-\sin 2x)\,dx=\left[-2\cos x+\frac{1}{2}\cos 2x\right]_0^{\pi}$$
$$=\left[\left(2+\frac{1}{2}\right)-\left(-2+\frac{1}{2}\right)\right]$$
$$=4$$

29. Use the region's symmetry:

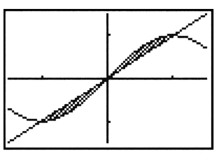
$$2\int_0^{\pi/3}(8\cos x-\sec^2 x)\,dx$$
$$=2[8\sin x-\tan x]_0^{\pi/3}$$
$$=2\left[\left(4\sqrt{3}-\sqrt{3}\right)-0\right]$$
$$=6\sqrt{3}$$

30. 

[–1.1, 1.1] by [–0.1, 1.1]

The curves intersect at $x = 0$ and $x = \pm 1$, but they do not cross at $x = 0$.

$$2\int_0^1 \left[1 - x^2 - \cos\left(\frac{\pi x}{2}\right)\right] dx$$
$$= 2\left[x - \frac{1}{3}x^3 - \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_0^1$$
$$= 2\left[\left(1 - \frac{1}{3} - \frac{2}{\pi}\right) - 0\right]$$
$$= \frac{4}{3} - \frac{4}{\pi} \approx 0.0601$$

31. 

[–1.5, 1.5] by [–1.5, 1.5]

The curves intersect at $x = 0$ and $x = \pm 1$. Use the area's symmetry:

$$2\int_0^1 \left[\sin\left(\frac{\pi x}{2}\right) - x\right] dx$$
$$= 2\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{1}{2}x^2\right]_0^1$$
$$= 2\left[-\frac{1}{2} - \left(-\frac{2}{\pi}\right)\right]$$
$$= \frac{4 - \pi}{\pi} \approx 0.273$$

32. Use the region's symmetry, and simplify before integrating:

$$2\int_0^{\pi/4} (\sec^2 x - \tan^2 x)\, dx$$
$$= 2\int_0^{\pi/4} [\sec^2 x - (\sec^2 x - 1)]\, dx$$
$$= 2\int_0^{\pi/4} dx$$
$$= 2[x]_0^{\pi/4}$$
$$= \frac{\pi}{2}$$

33. Use the region's symmetry:

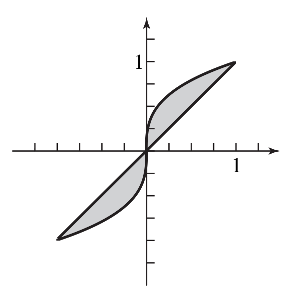
$$2\int_0^{\pi/4} (\tan^2 y + \tan^2 y)\, dy = 4\int_0^{\pi/4} \tan^2 y\, dy$$
$$= 4[\tan y - y]_0^{\pi/4}$$
$$= 4\left[\left(1 - \frac{\pi}{4}\right) - 0\right]$$
$$= 4 - \pi \approx 0.858$$

34. $$\int_0^{\pi/2} 3\sin y\sqrt{\cos y}\, dy = 3\left[-\frac{2}{3}(\cos y)^{3/2}\right]_0^{\pi/2}$$
$$= 3\left[0 - \left(-\frac{2}{3}\right)\right]$$
$$= 2$$

35. $$\int_{-3}^1 \sqrt{x+3}\, dx - \int_0^1 (2x)\, dx$$
$$= \frac{2}{3}(x+3)^{3/2}\Big|_3^1 - x^2\Big|_0^1$$
$$= \frac{2}{3}(1+3)^{3/2} - \frac{2}{3}(-3+3)^{3/2} - (1^2 - 0^2)$$
$$\approx 4.333$$

36. $$\int_{-2}^1 (4 - x^2)\, dx - \int_0^1 (3x)\, dx$$
$$= \left(4x - \frac{x^3}{3}\right)\Big|_{-2}^1 - \frac{3x^2}{2}\Big|_0^1$$
$$= 4 - \frac{1^3}{3} - \left(4(-2) - \frac{-2^3}{3}\right) - \left(\frac{3(1)^2}{2} - 0\right)$$
$$= \frac{15}{2}$$

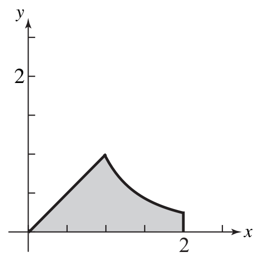
37. Solve for x: $x = y^3$ and $x = y$.



The curves intersect at $y = 0$ and $y = \pm 1$. Both functions are odd. Use the area's symmetry:

$$2\int_0^1 (y - y^3)\, dy = 2\left[\frac{1}{2}y^2 - \frac{1}{4}y^4\right]_0^1 = \frac{1}{2}$$

38.



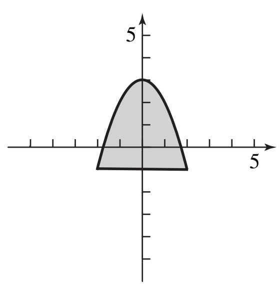
$y = x$ and $y = \dfrac{1}{x^2}$ intersect at $x = 1$. Integrate in two parts:

$$\int_0^1 x\,dx + \int_1^2 \frac{1}{x^2}\,dx = \left[\frac{1}{2}x^2\right]_0^1 + \left[-\frac{1}{x}\right]_1^2$$
$$= \frac{1}{2} + \left[-\frac{1}{2} - (-1)\right]$$
$$= 1$$

39. The curves intersect when $\sin x = \cos x$, i.e., at $x = \dfrac{\pi}{4}$.

$$\int_0^{\pi/4} (\cos x - \sin x)\,dx = [\sin x + \cos x]_0^{\pi/4}$$
$$= \sqrt{2} - 1$$

40.



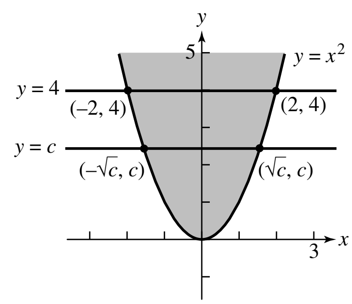
(a) The curves intersect at $x = \pm 2$.
Use the region's symmetry:

$$2\int_0^2 (3 - x^2 + 1)\,dx = 2\int_0^2 (4 - x^2)\,dx$$
$$= 2\left[4x - \frac{1}{3}x^3\right]_0^2$$
$$= 2\left[\left(8 - \frac{8}{3}\right) - 0\right]$$
$$= \frac{32}{3}$$

(b) Solve $y = 3 - x^2$ for x: $x = \pm\sqrt{3 - y}$. The y-intercepts are -1 and 3.

$$\int_{-1}^3 2\sqrt{3 - y}\,dy = 2\left[-\frac{2}{3}(3 - y)^{3/2}\right]_{-1}^3$$
$$= 2\left[0 - \left(-\frac{16}{3}\right)\right]$$
$$= \frac{32}{3}$$

41. (a)



If $y = x^2 = c$, then $x = \pm\sqrt{c}$. So the points are $\left(-\sqrt{c},\ c\right)$ and $\left(\sqrt{c},\ c\right)$.

(b) The two areas in Quadrant I, where $x = \sqrt{y}$, are equal:

$$\int_0^c \sqrt{y}\,dy = \int_c^4 \sqrt{y}\,dy$$
$$\int_0^{20} \sqrt{1 + \left(\frac{3\pi}{20}\cos\frac{3\pi}{20}x\right)^2}\,dx,$$
$$\frac{2}{3}c^{3/2} = \frac{2}{3}4^{3/2} - \frac{2}{3}c^{3/2}$$
$$2c^{3/2} = 8$$
$$c^{3/2} = 4$$
$$c = 4^{2/3} = 2^{4/3}$$

(c) Divide the upper right section into a (4 – c)-by-$\sqrt{c}$ rectangle and a leftover portion:

$$\int_0^{\sqrt{c}}(c-x^2)\,dx=(4-c)\sqrt{c}+\int_{\sqrt{c}}^2(4-x^2)\,dx$$

$$\left[cx-\frac{1}{3}x^3\right]_0^{\sqrt{c}}=4\sqrt{c}-c^{3/2}+\left[4x-\frac{1}{3}x^3\right]_{\sqrt{c}}^2$$

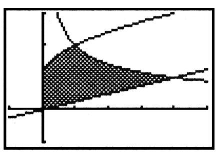
$$c^{3/2}-\frac{1}{3}c^{3/2}=4\sqrt{c}-c^{3/2}+\left[\left(8-\frac{8}{3}\right)-\left(4\sqrt{c}-\frac{1}{3}c^{3/2}\right)\right]$$

$$\frac{2}{3}c^{3/2}=4\sqrt{c}-c^{3/2}+\frac{16}{3}-4\sqrt{c}+\frac{1}{3}c^{3/2}$$

$$\frac{4}{3}c^{3/2}=\frac{16}{3}$$

$$c^{3/2}=4$$

$$c=4^{2/3}=2^{4/3}$$

42. 

[–1, 5] by [–1, 3]

The key intersection points are at $x = 0$, $x = 1$ and $x = 4$. Integrate in two parts:

$$\int_0^1\left(1+\sqrt{x}-\frac{x}{4}\right)dx+\int_1^4\left(\frac{2}{\sqrt{x}}-\frac{x}{4}\right)dx$$

$$=\left[x+\frac{2}{3}x^{3/2}-\frac{x^2}{8}\right]_0^1+\left[4\sqrt{x}-\frac{x^2}{8}\right]_1^4$$

$$=\left(1+\frac{2}{3}-\frac{1}{8}\right)+\left[(8-2)-\left(4-\frac{1}{8}\right)\right]$$

$$=\frac{11}{3}$$

43. First find the two areas.

For the triangle, $\frac{1}{2}(2a)(a^2)=a^3$

For the parabola, $2\int_0^a(a^2-x^2)\,dx=2\left[a^2x-\frac{1}{3}x^3\right]_0^a=\frac{4}{3}a^3$

The ratio, then, is $\dfrac{a^3}{\frac{4}{3}a^3}=\dfrac{3}{4}$, which remains constant as a approaches zero.

44. $\int_a^b[2f(x)-f(x)]\,dx=\int_a^b f(x)\,dx$, which we already know equals 4.

45. Neither; both integrals come out as zero because the –1-to-0 and 0-to-1 portions of the integrals cancel each other.

46. Sometimes true, namely when $dA = [f(x) - g(x)]\,dx$ is always nonnegative. This happens when $f(x) \geq g(x)$ over the entire interval.

47. Solve $\dfrac{2x}{x^2+1} = x^3$.

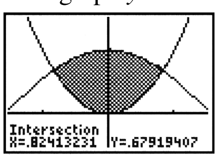
The curves intersect at $x = 0$ and $x = \pm 1$. Both functions are odd. Use the area's symmetry:

$$2\int_0^1 \left(\frac{2x}{x^2+1} - x^3\right) dx = 2\left[\ln(x^2+1) - \frac{1}{4}x^4\right]_0^1$$
$$= 2\ln 2 - \frac{1}{2}$$
$$= \ln 4 - \frac{1}{2} \approx 0.886$$

48. Solve $\sin x = x^3$. The curves intersect at $x = 0$ and $x \approx \pm 0.9286$. Both functions are odd. Use symmetry.

$$2\int_0^{0.9286} (\sin x - x^3) = 2\left(-\cos x - \frac{1}{4}x^4\right)\Bigg|_0^{0.296}$$
$$\approx 0.4303$$

49. First graph $y = \cos x$ and $y = x^2$.



[−1.5, 1.5] by [−0.5, 1.5]

The curves intersect at $x \approx \pm 0.8241$. Use NINT to find $2\int_0^{0.8241} (\cos x - x^2)\, dx \approx 1.0948$.

Multiplying both functions by k will not change the x-value of any intersection point, so the area condition to be met is

$$2 = 2\int_0^{0.8241} (k\cos x - kx^2)\, dx$$
$$\Rightarrow 2 = k \cdot 2\int_0^{0.8241} (\cos x - x^2)\, dx$$
$$\Rightarrow 2 \approx k(1.0948)$$
$$\Rightarrow k \approx 1.8269.$$

50. True; 36 is the value of the appropriate integral.

51. False; it is $\int_0^{0.739} (\cos x - x)\, dx$.

52. A

53. E; $\int_0^3 (x^2 - (-x))\, dx = \left(\dfrac{x^3}{3} + \dfrac{x^2}{2}\right)\Bigg|_0^3 = \dfrac{27}{2}$

54. B; the curves intersect at $x \approx 1.139$. Use NINT to find $\int_0^{1.139} \left(e^{-x^2} - (-\sin 3x)\right) dx \approx 1.445$

55. A; $\int_1^2 \left(e^x - \dfrac{1}{x}\right) dx = (e^x - \ln x)\Big|_1^2 = e^2 - \ln 2 - e$

56. **(a)** Solve for y:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$
$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$

(b) $\int_{-a}^{a} \left[b\sqrt{1 - \dfrac{x^2}{a^2}} - \left(-b\sqrt{1 - \dfrac{x^2}{a^2}}\right)\right] dx$ or

$2\int_{-a}^{a} b\sqrt{1 - \dfrac{x^2}{a^2}}\, dx$ or $4\int_0^a b\sqrt{1 - \dfrac{x^2}{a^2}}\, dx$

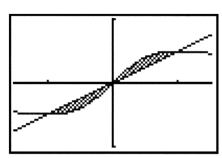
(c) Answers may vary.

(d, e) $2\int_{-a}^{a} b\sqrt{1 - \dfrac{x^2}{a^2}}\, dx$

$$= 2b\left[\frac{x}{2}\sqrt{1 - \frac{x^2}{a^2}} + \frac{a}{2}\sin^{-1}\frac{x}{a}\right]_{-a}^{a}$$
$$= 2b\left[\frac{a}{2}\sin^{-1}(1) - \frac{a}{2}\sin^{-1}(-1)\right]$$
$$= \pi ab$$

57. By hypothesis, $f(x) - g(x)$ is the same for each region, where $f(x)$ and $g(x)$ represent the upper and lower edges. But then Area $= \int_a^b [f(x) - g(x)]\, dx$ will be the same for each.

58. The curves are shown for $m = \dfrac{1}{2}$:



[−1.5, 1.5] by [−1, 1]

In general, the intersection points are where $\dfrac{x}{x^2+1} = mx$, which is where

$x = 0$ or else $x = \pm\sqrt{\dfrac{1}{m} - 1}$. Then, because of

symmetry, the area is

$$2\int_0^{\sqrt{(1/m)-1}}\left(\frac{x}{x^2+1}-mx\right)dx$$
$$=2\left[\frac{1}{2}\ln(x^2+1)-\frac{1}{2}mx^2\right]_0^{\sqrt{(1/m)-1}}$$
$$=\ln\left(\frac{1}{m}-1+1\right)-m\left(\frac{1}{m}-1\right)$$
$$=m-\ln(m)-1.$$

Section 8.3 Volumes (pp. 403–414)

Exploration 1 Volume by Cylindrical Shells

1. Its height is $f(x_k)=3x_k-x_k^2$.

2. Unrolling the cylinder, the circumference becomes one dimension of a rectangle, and the height becomes the other. The thickness Δx is the third dimension of a slab with dimensions $2\pi(x_k+1)$ by $3x_k-x_k{}^2$ by Δx. The volume is obtained by multiplying the dimensions together.

3. The limit is the definite integral $\int_0^3 2\pi(x+1)(3x-x^2)\,dx$

4. $\frac{45\pi}{2}$

Exploration 2 Surface Area

1. $\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

   The limit will exist if f and f' are continuous on the interval $[a, b]$.

2. $y=\sin x$, so $\frac{dy}{dx}=\cos x$ and

$$\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$=\int_0^\pi 2\pi\sin x\sqrt{1+\cos^2 x}\,dx$$
$$\approx 14.424.$$

3. $y=\sqrt{x}$, so $\frac{dy}{dx}=\frac{1}{2\sqrt{x}}$ and

$$\int_0^4 2\pi\sqrt{x}\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,dx\approx 36.177.$$

Quick Review 8.3

1. x^2

2. $s=\frac{x}{\sqrt{2}}$, so Area $=s^2=\frac{x^2}{2}$.

3. $\frac{1}{2}\pi r^2$ or $\frac{\pi x^2}{2}$

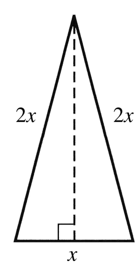
4. $\frac{1}{2}\pi\left(\frac{d}{2}\right)^2$ or $\frac{\pi x^2}{8}$

5. $b=x$ and $h=\frac{\sqrt{3}}{2}x$, so Area $=\frac{1}{2}bh=\frac{\sqrt{3}}{4}x^2$.

6. $b=h=x$, so Area $=\frac{1}{2}bh=\frac{x^2}{2}$.

7. $b=h=\frac{x}{\sqrt{2}}$, so Area $=\frac{1}{2}bh=\frac{x^2}{4}$

8.



$b=x$ and

$$h=\sqrt{(2x^2)-\left(\frac{1}{2}x\right)^2}=\frac{\sqrt{15}}{2}x,\text{ so}$$

Area $=\frac{1}{2}bh=\frac{\sqrt{15}}{4}x^2$.

9. This is a 3-4-5 right triangle. $b=4x$, $h=3x$, and Area $=\frac{1}{2}bh=6x^2$.

10. The hexagon contains six equilateral triangles with sides of length x, so from Exercise 5,

Area $= 6\left(\dfrac{\sqrt{3}}{4}x^2\right) = \dfrac{3\sqrt{3}}{2}x^2.$

Section 8.3 Exercises

1. In each case, the width of the cross section is $w = 2\sqrt{1-x^2}$.

(a) $A = \pi r^2$, where $r = \dfrac{w}{2}$, so

$$A(x) = \pi\left(\frac{w}{2}\right)^2 = \pi(1-x^2).$$

(b) $A = s^2$, where $s = w$, so $A(x) = w^2 = 4(1-x^2)$.

(c) $A = s^2$, where $s = \dfrac{w}{\sqrt{2}}$, so

$$A(x) = \left(\frac{w}{\sqrt{2}}\right)^2 = 2(1-x^2).$$

(d) $A = \dfrac{\sqrt{3}}{4}w^2$ (see Quick Review Exercise 5), so

$$A(x) = \frac{\sqrt{3}}{4}\left(2\sqrt{1-x^2}\right)^2 = \sqrt{3}(1-x^2).$$

2. In each case, the width of the cross section is $w = 2\sqrt{x}$.

(a) $A = \pi r^2$, where $r = \dfrac{w}{2}$, so

$$A(x) = \pi\left(\frac{w}{2}\right)^2 = \pi x.$$

(b) $A = s^2$, where $s = w$, so $A(x) = w^2 = 4x$.

(c) $A = s^2$, where $s = \dfrac{w}{\sqrt{2}}$, so

$$A(x) = \left(\frac{w}{\sqrt{2}}\right)^2 = 2x.$$

(d) $A = \dfrac{\sqrt{3}}{4}w^2$ (see Quick Review Exercise 5), so $A(x) = \dfrac{\sqrt{3}}{4}\left(2\sqrt{x}\right)^2 = \sqrt{3}x.$

3. A cross section has width $w = 2\sqrt{x}$ and area $A(x) = s^2 = \left(\dfrac{w}{\sqrt{2}}\right)^2 = 2x.$ The volume is

$$\int_0^4 2x\,dx = [x^2]_0^4 = 16.$$

4. A cross section has width $w = (2-x^2) - x^2 = 2 - 2x^2$ and area $A(x) = \pi r^2 = \pi\left(\dfrac{w}{2}\right)^2 = \pi(1-x^2)^2.$ The volume is

$$\begin{aligned}\int_{-1}^{1} (1-x^2)^2\,dx &= \pi\int_{-1}^{1} (x^4 - 2x^2 + 1)\,dx \\ &= \pi\left[\frac{1}{5}x^5 - \frac{2}{3}x^3 + x\right]_{-1}^{1} \\ &= \frac{16}{15}\pi.\end{aligned}$$

5. The cross section has width $w = 2\sqrt{1-x^2}$ and area $A(x) = s^2 = w^2 = 4(1-x^2).$ The volume is

$$\begin{aligned}\int_{-1}^{1} 4(1-x^2)\,dx &= 4\int_{-1}^{1} (1-x^2)\,dx \\ &= 4\left[x - \frac{1}{3}x^3\right]_{-1}^{1} \\ &= \frac{16}{3}.\end{aligned}$$

6. A cross section has width $w = 2\sqrt{1-x^2}$ and area $A(x) = s^2 = \left(\dfrac{w}{\sqrt{2}}\right)^2 = 2(1-x^2).$ The volume is

$$\begin{aligned}\int_{-1}^{1} 2(1-x^2)\,dx &= 2\int_{-1}^{1} (1-x^2)\,dx \\ &= 2\left[x - \frac{1}{3}x^3\right]_{-1}^{1} \\ &= \frac{8}{3}.\end{aligned}$$

7. The solid is a right circular cone of radius 1 and height 2.

$V = \frac{1}{3}Bh = \frac{1}{3}(\pi r^2)h = \frac{1}{3}(\pi \cdot 1^2)\cdot 2 = \frac{2}{3}\pi$

Using integration: A cross section has radius $\left(1-\frac{1}{2}x\right)$ and area $A(x) = \pi\left(1-\frac{1}{2}x\right)^2$. The volume is $V = \int_0^2 \pi\left(1-\frac{1}{2}x\right)^2 dx$

$$= \pi\int_0^2\left(\frac{x^2}{4}-x+1\right)dx$$
$$= \pi\left[\frac{x^3}{12}-\frac{x^2}{2}+x\right]_0^2$$
$$= \frac{2}{3}\pi.$$

8. The solid is a right circular cone of radius 3 and height 2.

$V = \frac{1}{3}Bh = \frac{1}{3}(\pi r^2)h = \frac{1}{3}(\pi \cdot 3^2)\cdot 2 = 6\pi$

Using integration: A cross section has radius $\left(\frac{3y}{2}\right)$ and area $A(y) = \pi\left(\frac{3y}{2}\right)^2$. The volume is $V = \int_0^2 \pi\left(\frac{3y}{2}\right)^2 dy$

$$= \frac{9}{4}\pi\int_0^2 y^2 dy$$
$$= \frac{9}{4}\pi\left[\frac{y^3}{3}\right]^2$$
$$= 6\pi.$$

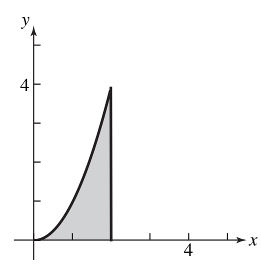
9. A cross section has radius $r = \tan\left(\frac{\pi}{4}y\right)$ and area $A(y) = \pi r^2 = \pi \tan^2\left(\frac{\pi}{4}y\right)$. The volume is $\int_0^1 \pi\tan^2\left(\frac{\pi}{4}y\right)dy = \pi\left[\frac{4}{\pi}\tan\left(\frac{\pi}{4}y\right)-y\right]_0^1$

$$= \pi\left(\frac{4}{\pi}-1\right)$$
$$= 4-\pi.$$

10. A cross section has radius $r = \sin x \cos x$ and area $A(x) = \pi r^2 = \pi \sin^2 x \cos^2 x$. The shaded region extends from $x = 0$ to where $\sin x \cos x$ drops back to 0, i.e., where $x = \frac{\pi}{2}$. Now, since $\cos 2x = 2\cos^2 x - 1$, we know $\cos^2 x = \frac{1+\cos 2x}{2}$ and since $\cos 2x = 1 - 2\sin^2 x$, we know $\sin^2 x = \frac{1-\cos 2x}{2}$.

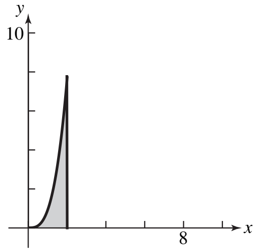
$$\int_0^{\pi/2} \pi \sin^2 x \cos^2 x\, dx$$
$$= \pi\int_0^{\pi/2}\frac{1-\cos 2x}{2}\cdot\frac{1+\cos 2x}{2}dx$$
$$= \frac{\pi}{4}\int_0^{\pi/2}(1-\cos^2 2x)\,dx$$
$$= \frac{\pi}{4}\int_0^{\pi/2}\sin^2 2x\,dx$$
$$= \frac{\pi}{4}\int_0^{\pi/2}\frac{1-\cos 4x}{2}dx$$
$$= \frac{\pi}{8}\int_0^{\pi/2}(1-\cos 4x)\,dx$$
$$= \frac{\pi}{8}\left[x-\frac{1}{4}\sin 4x\right]_0^{\pi/2}$$
$$= \frac{\pi}{8}\left[\left(\frac{\pi}{2}-0\right)-0\right]$$
$$= \frac{\pi^2}{16}.$$

11.



A cross section has radius $r = x^2$ and area $A(x) = \pi r^2 = \pi x^4$. The volume is $\int_0^2 \pi x^4\, dx = \pi\left[\frac{1}{5}x^5\right]_0^2 = \frac{32\pi}{5}.$

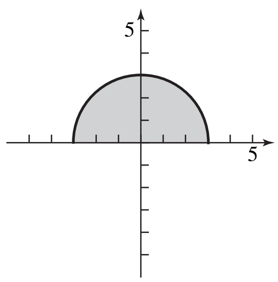
12.



A cross section has radius $r = x^3$ and area $A(x) = \pi r^2 = \pi x^6$. The volume is

$$\int_0^2 \pi x^6\, dx = \pi\left[\frac{1}{7}x^7\right]_0^2 = \frac{128\pi}{7}.$$

13.

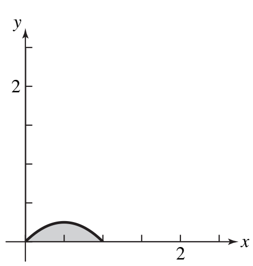


The solid is a sphere of radius $r = 3$. The volume is $\frac{4}{3}\pi r^3 = 36\pi$.

Using integration: A cross section has radius $\sqrt{9-x^2}$ and area $A(y) = \pi\left(\sqrt{9-x^2}\right)^2$. The volume is $V = \int_{-3}^{3}\pi\left(\sqrt{9-x^2}\right)^2 dx$

$$= \pi\int_{-3}^{3}(9-x^2)\,dx$$
$$= \pi\left[9x - \frac{1}{3}x^3\right]_{-3}^{3}$$
$$= 36\pi.$$

14.

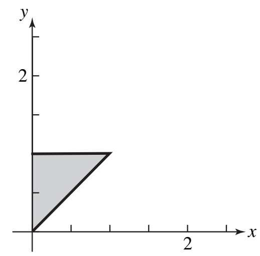


The parabola crosses the line $y = 0$ when $x - x^2 = x(1-x) = 0$, i.e., when $x = 0$ or $x = 1$. A cross section has radius $r = x - x^2$ and area $A(x) = \pi r^2 = \pi(x-x^2)^2 = \pi(x^2 - 2x^3 + x^4)$.

The volume is

$$\int_0^1 \pi\left(x^2 - 2x^3 + x^4\right)dx$$
$$= \pi\left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_0^1$$
$$= \frac{\pi}{30}.$$

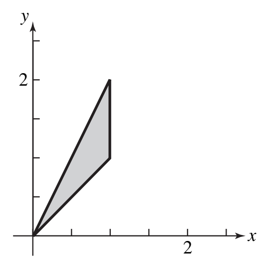
15.



Use cylindrical shells: A shell has radius y and height y. The volume is

$$\int_0^1 2\pi(y)(y)\,dy = 2\pi\left[\frac{1}{3}y^3\right]_0^1 = \frac{2}{3}\pi.$$

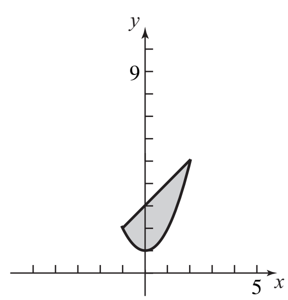
16.



Use washer cross sections: A washer has inner radius $r = x$, outer radius $R = 2x$, and area $A(x) = \pi(R^2 - r^2) = 3\pi x^2$. The volume is

$$\int_0^1 3\pi x^2\, dx = 3\pi\left[\frac{1}{3}x^3\right]_0^1 = \pi.$$

17.



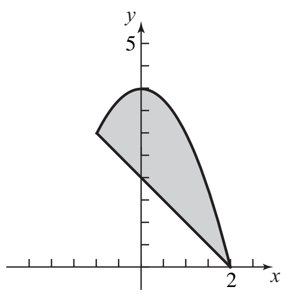
The curves intersect when $x^2 + 1 = x + 3$, which is when $x^2 - x - 2 = (x-2)(x+1) = 0$, i.e., when $x = -1$ or $x = 2$. Use washer cross sections: a washer has inner radius $r = x^2 + 1$, outer radius $R = x + 3$, and area

$$\begin{aligned}A(x) &= \pi(R^2 - r^2)\\ &= \pi\left[(x+3)^2 - (x^2+1)^2\right]\\ &= \pi(-x^4 - x^2 + 6x + 8).\end{aligned}$$

The volume is

$$\int_{-1}^{2} \pi(-x^4 - x^2 + 6x + 8)\,dx$$
$$= \pi\left[-\frac{1}{5}x^5 - \frac{1}{3}x^3 + 3x^2 + 8x\right]_{-1}^{2}$$
$$= \pi\left[\left(-\frac{32}{5} - \frac{8}{3} + 12 + 16\right) - \left(\frac{1}{5} + \frac{1}{3} + 3 - 8\right)\right]$$
$$= \frac{117\pi}{5}$$

18.



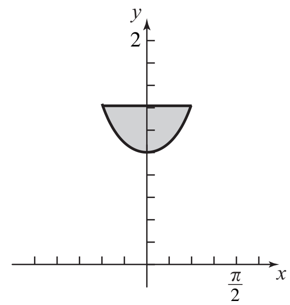
The curves intersect when $4 - x^2 = 2 - x$, which is when $x^2 - x - 2 = (x-2)(x+1) = 0$, i.e., when $x = -1$ or $x = 2$. Use washer cross sections: a washer has inner radius $r = 2 - x$, outer radius $R = 4 - x^2$, and area

$$\begin{aligned}A(x) &= \pi(R^2 - r^2)\\ &= \pi\left[(4-x^2)^2 - (2-x)^2\right]\\ &= \pi(12 + 4x - 9x^2 + x^4).\end{aligned}$$

The volume is

$$\int_{-1}^{2} \pi(12 + 4x - 9x^2 + x^4)\,dx$$
$$= \pi\left[12x + 2x^2 - 3x^3 + \frac{1}{5}x^5\right]_{-1}^{2}$$
$$= \pi\left[\left(24 + 8 - 24 + \frac{32}{5}\right) - \left(-12 + 2 + 3 - \frac{1}{5}\right)\right]$$
$$= \frac{108\pi}{5}.$$

19.

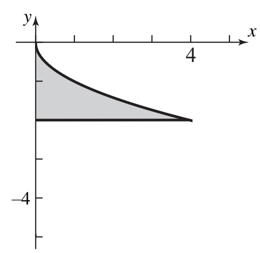


Use washer cross sections: a washer has inner radius $r = \sec x$. outer radius $R = \sqrt{2}$, and area $\int_0^{1.8933} 2\pi(x)\left(3^{1-x^2} - \frac{x^2-3}{10}\right)dx$,

The volume is

$$\int_{-\pi/4}^{\pi/4} \pi(2 - \sec^2 x)\,dx$$
$$= \pi[2x - \tan x]_{-\pi/4}^{\pi/4}$$
$$= \pi\left[\left(\frac{\pi}{2} - 1\right) - \left(-\frac{\pi}{2} + 1\right)\right]$$
$$= \pi^2 - 2\pi.$$

20.

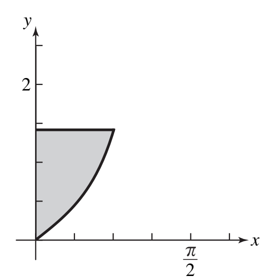


The curves intersect where $-\sqrt{x} = -2$, which is where $x = 4$. Use washer cross sections: a washer has inner radius $r = \sqrt{x}$, outer radius $R = 2$, and area $A(x) = \pi(R^2 - r^2) = \pi(4 - x)$.

The volume is

$$\int_0^4 \pi(4 - x)\,dx = \pi\left[4x - \frac{1}{2}x^2\right]_0^4 = 8\pi$$

21.

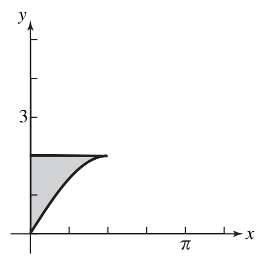


The curves intersect at $x = \frac{\pi}{4}$. A cross section

has radius $r = \sqrt{2} - \sec x \tan x$ and area $A(x) = \pi r^2 = \pi\left(\sqrt{2} - \sec x \tan x\right)^2$. The volume is

$$\int_0^{\pi/4} \pi\left(\sqrt{2} - \sec x \tan x\right)^2 dx$$
$$= \int_0^{\pi/4} \pi\left(2 - 2\sqrt{2}\sec x \tan x + \sec^2 x \tan^2 x\right) dx$$
$$= \pi\left[2x - 2\sqrt{2}\sec x + \frac{1}{3}\tan^3 x\right]_0^{\pi/4}$$
$$= \pi\left[\left(\frac{\pi}{2} - 4 + \frac{1}{3}\right) - \left(-2\sqrt{2}\right)\right]$$
$$\approx 2.301$$

22.

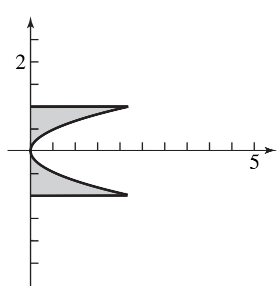


The curve and horizontal line intersect at $x = \frac{\pi}{2}$. A cross section has radius $2 - 2\sin x$ and area $A(x) = \pi r^2 = 4\pi(1 - \sin x)^2$
$= 4\pi(1 - 2\sin x + \sin^2 x)$.

The volume is

$$\int_0^{\pi/2} 4\pi(1 - 2\sin x + \sin^2 x)\, dx$$
$$= \int_0^{\pi/2} 4\pi\left(1 - 2\sin x + \frac{1 - \cos 2x}{2}\right) dx$$
$$= 4\pi\left[\frac{3}{2}x + 2\cos x - \frac{1}{4}\sin 2x\right]_0^{\pi/2}$$
$$= 4\pi\left(\frac{3\pi}{4} - 2\right)$$
$$= \pi(3\pi - 8)$$

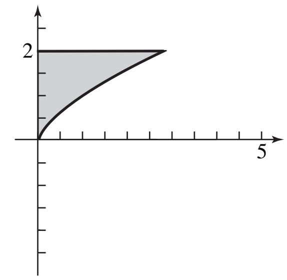
23.



A cross section has radius $r = \sqrt{5}y^2$ and area $A(y) = \pi r^2 = 5\pi y^4$.

The volume is $\int_{-1}^{1} 5\pi y^4\, dy = \pi[y^5]_{-1}^{1} = 2\pi$.

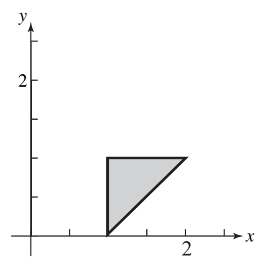
24.



A cross section has radius $r = y^{3/2}$ and area $A(y) = \pi r^2 = \pi y^3$. The volume is

$$\int_0^2 \pi y^3\, dy = \pi\left[\frac{1}{4}y^4\right]_0^2 = 4\pi.$$

25.



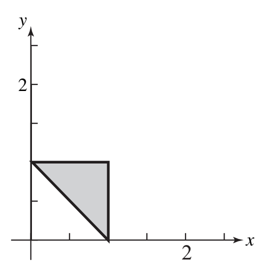
Use washer cross sections. A washer has inner radius $r = 1$. Outer radius $R = y + 1$, and area

$$A(y) = \pi(R^2 - r^2)$$
$$= \pi[(y+1)^2 - 1]$$
$$= \pi(y^2 + 2y).$$

The volume is

$$\int_0^1 \pi(y^2 + 2y)\, dy = \pi\left[\frac{1}{3}y^3 + y^2\right]_0^1 = \frac{4}{3}\pi.$$

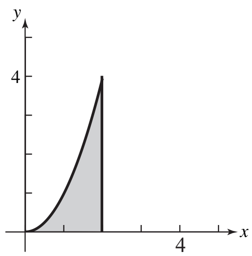
26.



Use cylindrical shells: a shell has radius x and height x. The volume is

$$\int_0^1 2\pi(x)(x)\, dx = 2\pi\left[\frac{1}{3}x^3\right]_0^1 = \frac{2}{3}\pi.$$

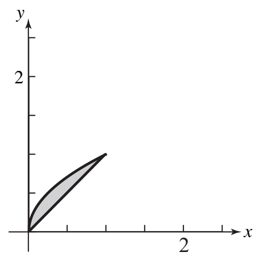
27.



Use cylindrical shells: A shell has radius x and height x^2. The volume is

$$\int_0^2 2\pi(x)(x^2)\,dx = 2\pi\left[\frac{1}{4}x^4\right]_0^2 = 8\pi.$$

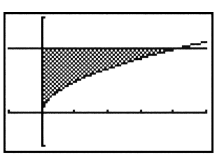
28.



The curves intersect at $x = 0$ and $x = 1$. Use cylindrical shells: a shell has radius x and height $\sqrt{x} - x$. The volume is

$$\int_0^1 2\pi(x)(\sqrt{x} - x)\,dx = 2\pi\left[\frac{2}{5}x^{5/2} - \frac{1}{3}x^3\right]_0^1 = \frac{2\pi}{15}.$$

29.



[–1, 5] by [–1, 3]

The curved and horizontal line intersect at (4, 2).

(a) Use washer cross sections: a washer has inner radius $r = \sqrt{x}$, outer radius $R = 2$, and area $A(x) = \pi(R^2 - r^2) = \pi(4 - x)$. The volume is

$$\int_0^4 \pi(4 - x)\,dx = \pi\left[4x - \frac{1}{2}x^2\right]_0^4 = 8\pi$$

(b) A cross section has radius $r = y^2$ and area $A(y) = \pi r^2 = \pi y^4$. The volume is

$$\int_0^2 \pi y^4\,dy = \pi\left[\frac{1}{5}y^5\right]_0^2 = \frac{32\pi}{5}.$$

(c) A cross section has radius $r = 2 - \sqrt{x}$ and area

$$\begin{aligned} A(x) &= \pi r^2 \\ &= \pi\left(2 - \sqrt{x}\right)^2 \\ &= \pi\left(4 - 4\sqrt{x} + x\right). \end{aligned}$$

The volume is

$$\begin{aligned} &\int_0^4 \pi(4 - 4\sqrt{x} + x)\,dx \\ &= \pi\left[4x - \frac{8}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^4 \\ &= \frac{8\pi}{3}. \end{aligned}$$

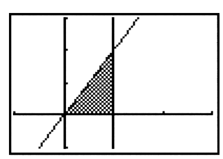
(d) Use washer cross sections: a washer has inner radius $r = 4 - y^2$, outer radius $R = 4$, and area

$$\begin{aligned} A(y) &= \pi(R^2 - r^2) \\ &= \pi[16 - (4 - y^2)^2] \\ &= \pi(8y^2 - y^4). \end{aligned}$$

The volume is

$$\int_0^2 \pi(8y^2 - y^4)\,dy = \pi\left[\frac{8}{3}y^3 - \frac{1}{5}y^5\right]_0^2 = \frac{224\pi}{15}$$

30.



[–1, 3] by [–1, 3]

The slanted and vertical lines intersect at (1, 2).

(a) The solid is a right circular cone of radius 1 and height 2. The volume is

$$\frac{1}{3}Bh = \frac{1}{3}(\pi r^2)h = \frac{1}{3}(\pi 1^2)2 = \frac{2}{3}\pi.$$

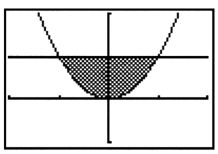
Using integration: A cross section has radius $1 - \frac{y}{2}$ and area

$A(y) = \pi r^2 = \pi\left(1 - \frac{y}{2}\right)^2$. The volume is

$$V = \int_0^2 \pi\left(1 - \frac{y}{2}\right)^2 dy$$
$$= \pi\int_0^2\left(1 - y + \frac{1}{4}y^2\right)dy$$
$$= \pi\left[y - \frac{1}{2}y^2 + \frac{1}{12}y^3\right]_0^2$$
$$= \frac{2}{3}\pi.$$

(b) Use cylindrical shells: shell has radius $2 - x$ and height $2x$. The volume is

$$\int_0^1 2\pi(2-x)(2x)\,dx = 4\pi\int_0^1 (2x - x^2)\,dx$$
$$= 4\pi\left[x^2 - \frac{1}{3}x^3\right]_0^1$$
$$= \frac{8\pi}{3}.$$

31. 

[–2, 2] by [–1, 2]

The curves intersect at $(\pm 1, 1)$.

(a) A cross section has radius $r = 1 - x^2$ and area

$A(x) = \pi r^2 = \pi(1 - x^2)^2 = \pi(1 - 2x^2 + x^4)$.

The volume is

$$\int_{-1}^1 \pi(1 - 2x^2 + x^4)\,dx$$
$$= \pi\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_{-1}^1$$
$$= \frac{16\pi}{15}.$$

(b) Use cylindrical shells: a shell has radius $2 - y$ and height $2\sqrt{y}$. The volume is

$$\int_0^1 2\pi(2-y)\left(2\sqrt{y}\right)dy$$
$$= 4\pi\int_0^1\left(2\sqrt{y} - y^{3/2}\right)dy$$
$$= 4\pi\left[\frac{4}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^1$$
$$= \frac{56\pi}{15}.$$

(c) Use cylindrical shells: a shell has radius $y + 1$ and height $2\sqrt{y}$. The volume is

$$\int_0^1 2\pi(y+1)\left(2\sqrt{y}\right)dy$$
$$= 4\pi\int_0^1\left(y^{3/2} + \sqrt{y}\right)dy$$
$$= 4\pi\left[\frac{2}{5}y^{5/2} + \frac{2}{3}y^{3/2}\right]_0^1$$
$$= \frac{64\pi}{15}.$$

32. (a) A cross section has radius $r = h\left(1 - \frac{x}{b}\right)$ and area $A(x) = \pi r^2 = \pi h^2\left(1 - \frac{x}{b}\right)^2$. The volume is

$$\int_0^b \pi h^2\left(1 - \frac{x}{b}\right)^2 dx = \pi h^2\left[-\frac{b}{3}\left(1 - \frac{x}{b}\right)^3\right]_0^b$$
$$= \frac{\pi}{3}bh^2.$$

(b) Use cylindrical shells: a shell has radius x and height $h\left(1 - \frac{x}{b}\right)$. The volume is

$$\int_0^b 2\pi(x)h\left(1 - \frac{x}{b}\right)dx = 2\pi h\int_0^b\left(x - \frac{x^2}{b}\right)dx$$
$$= 2\pi h\left[\frac{1}{2}x^2 - \frac{x^3}{3b}\right]_0^b$$
$$= \frac{\pi}{3}b^2h.$$

33. A shell has height $12(y^2 - y^3)$.

(a) A shell has radius y. The volume is

$$\int_0^1 2\pi(y)12(y^2 - y^3)\,dy$$
$$= 24\pi\int_0^1 (y^3 - y^4)\,dy$$
$$= 24\pi\left[\frac{1}{4}y^4 - \frac{1}{5}y^5\right]_0^1$$
$$= \frac{6\pi}{5}.$$

(b) A shell has radius $1 - y$. The volume is

$$\int_0^1 2\pi(1-y)12(y^2 - y^3)\,dy$$
$$= 24\pi\int_0^1 (y^4 - 2y^3 + y^2)\,dy$$
$$= 24\pi\left[\frac{1}{5}y^5 - \frac{1}{2}y^4 + \frac{1}{3}y^3\right]_0^1$$
$$= \frac{4\pi}{5}.$$

(c) A shell has radius $\frac{8}{5} - y$. The volume is

$$\int_0^1 2\pi\left(\frac{8}{5} - y\right)12(y^2 - y^3)dy$$
$$= 24\pi\int_0^1\left(y^4 - \frac{13}{5}y^3 + \frac{8}{5}y^2\right)dy$$
$$= 24\pi\left[\frac{1}{5}y^5 - \frac{13}{20}y^4 + \frac{8}{15}y^3\right]_0^1$$
$$= 2\pi.$$

(d) A shell has radius $y + \frac{2}{5}$. The volume is

$$\int_0^1 2\pi\left(y + \frac{2}{5}\right)12(y^2 - y^3)dy$$
$$= 24\pi\int_0^1\left(-y^4 + \frac{3}{5}y^3 + \frac{2}{5}y^2\right)dx$$
$$= 24\pi\left[-\frac{1}{5}y^5 + \frac{3}{20}y^4 + \frac{2}{15}y^3\right]_0^1$$
$$= 2\pi.$$

34. A shell has height

$$\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2}\right) = y^2 - \frac{y^4}{4}.$$

(a) A shell has radius y. The volume is

$$\int_0^2 2\pi(y)\left(y^2 - \frac{y^4}{4}\right)dy$$
$$= 2\pi\left[\frac{1}{4}y^4 - \frac{1}{24}y^6\right]_0^2$$
$$= \frac{8\pi}{3}.$$

(b) A shell has radius $2 - y$. The volume is

$$\int_0^2 2\pi(2-y)\left(y^2 - \frac{y^4}{4}\right)dy$$
$$= 2\pi\int_0^2\left(\frac{y^5}{4} - \frac{y^4}{2} - y^3 + 2y^2\right)dy$$
$$= 2\pi\left[\frac{1}{24}y^6 - \frac{1}{10}y^5 - \frac{1}{4}y^4 + \frac{2}{3}y^3\right]_0^2$$
$$= \frac{8\pi}{5}.$$

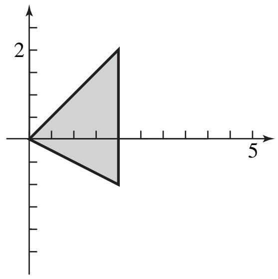
(c) A shell has radius $5 - y$. The volume is

$$\int_0^2 2\pi(5-y)\left(y^2 - \frac{y^4}{4}\right)dy$$
$$= 2\pi\int_0^2\left(\frac{y^5}{4} - \frac{5y^4}{4} - y^3 + 5y^2\right)dy$$
$$= 2\pi\left[\frac{1}{24}y^6 - \frac{1}{4}y^5 - \frac{1}{4}y^4 + \frac{5}{3}y^3\right]_0^2$$
$$= 8\pi.$$

(d) A shell has radius $y + \frac{5}{8}$. The volume is

$$\int_0^2 2\pi\left(y + \frac{5}{8}\right)\left(y^2 - \frac{y^4}{4}\right)dy$$
$$= 2\pi\int_0^2\left(-\frac{y^5}{4} - \frac{5y^4}{32} + y^3 + \frac{5y^2}{8}\right)dy$$
$$= 2\pi\left[-\frac{1}{24}y^6 - \frac{1}{32}y^5 + \frac{1}{4}y^4 + \frac{5}{24}y^3\right]_0^2$$
$$= 4\pi.$$

35.



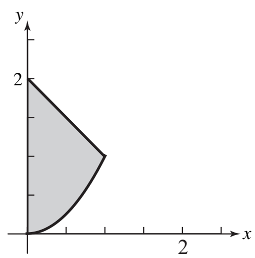
A shell has radius x and height

$x-\left(-\frac{x}{2}\right)=\frac{3}{2}x.$

The volume is

$\int_0^2 2\pi(x)\left(\frac{3}{2}x\right)dx=\pi[x^3]_0^2=8\pi.$

36.



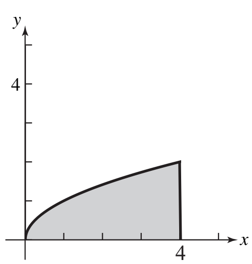
$x^2=2-x$ at $x=1$. A shell has radius x and height $2-x-x^2$. The volume is

$\int_0^1 2\pi(x)(2-x-x^2)dx$

$=2\pi\left[x^2-\frac{1}{3}x^3-\frac{1}{4}x^4\right]_0^1$

$=\frac{5\pi}{6}.$

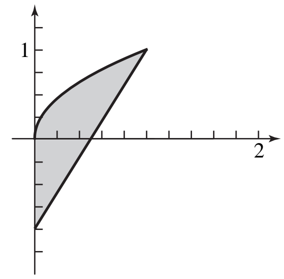
37.



A shell has radius x and height $\sqrt{x}$. The volume is

$\int_0^4 2\pi(x)(\sqrt{x})\,dx=2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4=\frac{128\pi}{5}.$

38.



The functions intersect where

$2x-1=\sqrt{x}$, i.e., at $x=1$.

A shell has radius x and height

$\sqrt{x}-(2x-1)=\sqrt{x}-2x+1$. The volume is

$\int_0^1 2\pi(x)(\sqrt{x}-2x+1)\,dx$

$=2\pi\int_0^1(x^{3/2}-2x^2+x)\,dx$

$=2\pi\left[\frac{2}{5}x^{5/2}-\frac{2}{3}x^3+\frac{1}{2}x^2\right]_0^1$

$=\frac{7\pi}{15}.$

39. A cross section has width $w=2\sqrt{\sin x}$.

(a) $A(x)=\frac{\sqrt{3}}{4}w^2=\sqrt{3}\sin x$, and

$V=\int_0^\pi\sqrt{3}\sin x\,dx$

$=\sqrt{3}\int_0^\pi\sin x\,dx$

$=\sqrt{3}[-\cos x]_0^\pi$

$=2\sqrt{3}.$

(b) $A(x)=s^2=w^2=4\sin x$, and

$V=\int_0^\pi 4\sin x\;dx$

$=4\int_0^\pi\sin x\;dx$

$=4[-\cos x]_0^\pi$

$=8.$

40. A cross section has width $w=\sec x-\tan x$.

(a) $A(x)=\pi r^2$

$=\pi\left(\frac{w}{2}\right)^2$

$=\frac{\pi}{4}(\sec x-\tan x)^2,$

and